Cramer's rule for some quaternion matrix equations.

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Abstract

Cramer's rules for some left, right and two-sided quaternion matrix equations are obtained within the framework of the theory of the column and row determinants.

Keywords: quaternion skew field, noncommutative determinant, inverse matrix, quaternion matrix equation, Cramer's rule.

MSC: 15A33, 15A15, 15A24.

1 Introduction

Quaternion matrix equations play important roles in both theoretical studies and numerical computations of quaternion application disciplines [1, 3, 8] and have been studied by many experts [5, 6, 7] in mainly by means of complex representation of quaternion matrices. In this paper we obtain Cramer's rule for some left, right and two-sided quaternion matrix equations within the framework of the theory of new matrix functionals over the quaternion skew field (the column and row determinants) introduced in [4]. In the first point we shortly cite some provisions from the theory of the column and row determinants which are necessary for the following. The theory of the column and row determinants of a quaternion matrix are considered completely in [4]. In the second point the Cramer rules for left, right and two-sided quaternion matrix equations are obtained.

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2 Elements of the theory of the column and row determinants.

Let $\mathrm{M}(n,\mathbb{H})$ be the ring of $n \times n$ quaternion matrices. By $\mathbb{H}^{m\times n}$ denote the set of all $m \times n$ matrices over the quaternion skew field $\mathbb{H}$ and by $\mathbb{H}_r^{m\times n}$ denote its subset of matrices of rank r. Suppose S_n is the symmetric group on the set $I_n = \{1,\ldots,n\}$.

Definition 2.1 *The ith row determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n,\mathbb{H})$ is defined by*

$$\mathrm{rdet}_i\mathbf{A} = \sum_{\sigma\in S_n} (-1)^{n-r} a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}}$$

for all $i = 1,\ldots,n$. The elements of the permutation σ are indices of each monomial. The left-ordered cycle notation of the permutation σ is written as follows,

$$\sigma = (i\, i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1})\, (i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}) \ldots (i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r}).$$

The index i opens the first cycle from the left and other cycles satisfy the following conditions, $i_{k_2} < i_{k_3} < \ldots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for all $t = 2,\ldots,r$ and $s = 1,\ldots,l_t$.

Definition 2.2 *The jth column determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n,\mathbb{H})$ is defined by*

$$\mathrm{cdet}_j\,\mathbf{A} = \sum_{\tau\in S_n} (-1)^{n-r} a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} i_{k_r}} \ldots a_{j\, j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}$$

for all $j = 1,\ldots,n$. The right-ordered cycle notation of the permutation $\tau \in S_n$ is written as follows,

$$\tau = (j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}) \ldots (j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2})\ (j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j).$$

The index j opens the first cycle from the right and other cycles satisfy the following conditions, $j_{k_2} < j_{k_3} < \ldots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for all $t = 2,\ldots,r$ and $s = 1,\ldots,l_t$.

Suppose $\mathbf{A}^{ij}$ denotes the submatrix of $\mathbf{A}$ obtained by deleting both the ith row and the jth column. Let $\mathbf{a}_{.j}$ be the jth column and $\mathbf{a}_{i.}$ be the ith row of $\mathbf{A}$. Suppose $\mathbf{A}_{.j}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its jth column with the column $\mathbf{b}$, and $\mathbf{A}_{i.}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its ith row with the row $\mathbf{b}$.

We note some properties of column and row determinants of a quaternion matrix $\mathbf{A} = (a_{ij})$, where $i \in I_n$, $j \in J_n$ and $I_n = J_n = \{1, \ldots, n\}$.

Proposition 2.1 *[4] If $b \in \mathbb{H}$, then $\mathrm{rdet}_i \mathbf{A}_{i.}(b \cdot \mathbf{a}_{i.}) = b \cdot \mathrm{rdet}_i \mathbf{A}$ for all $i = 1, \ldots, n$.*

Proposition 2.2 *[4] If $b \in \mathbb{H}$, then $\mathrm{cdet}_j \mathbf{A}_{.j}(\mathbf{a}_{.j} b) = \mathrm{cdet}_j \mathbf{A} b$ for all $j = 1, \ldots, n$.*

Proposition 2.3 *[4] If for $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ there exists $t \in I_n$ such that $a_{tj} = b_j + c_j$ for all $j = 1, \ldots, n$, then*

$$\begin{array}{l} \mathrm{rdet}_i \mathbf{A} = \mathrm{rdet}_i \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{rdet}_i \mathbf{A}_{t.}(\mathbf{c}), \\ \mathrm{cdet}_i \mathbf{A} = \mathrm{cdet}_i \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{cdet}_i \mathbf{A}_{t.}(\mathbf{c}), \end{array}$$

where $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{c} = (c_1, \ldots, c_n)$ and for all $i = 1, \ldots, n$.

Proposition 2.4 *[4] If for $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ there exists $t \in J_n$ such that $a_{i\,t} = b_i + c_i$ for all $i = 1, \ldots, n$, then*

$$\begin{array}{l} \mathrm{rdet}_j \mathbf{A} = \mathrm{rdet}_j \mathbf{A}_{.\,t}(\mathbf{b}) + \mathrm{rdet}_j \mathbf{A}_{.\,t}(\mathbf{c}), \\ \mathrm{cdet}_j \mathbf{A} = \mathrm{cdet}_j \mathbf{A}_{.\,t}(\mathbf{b}) + \mathrm{cdet}_j \mathbf{A}_{.\,t}(\mathbf{c}), \end{array}$$

where $\mathbf{b} = (b_1, \ldots, b_n)^T$, $\mathbf{c} = (c_1, \ldots, c_n)^T$ and for all $j = 1, \ldots, n$.

The following lemmas enable us to expand $\mathrm{rdet}_i \mathbf{A}$ by cofactors along the ith row and $\mathrm{cdet}_j \mathbf{A}$ along the jth column respectively for all $i, j = 1, \ldots, n$.

Lemma 2.1 *[4] Let R_{ij} be the right ij-th cofactor of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, that is, $\mathrm{rdet}_i \mathbf{A} = \sum_{j=1}^{n} a_{ij} \cdot R_{ij}$ for all $i = 1, \ldots, n$. Then*

$$R_{ij} = \begin{cases} -\mathrm{rdet}_j \mathbf{A}^{ii}_{.j}(\mathbf{a}_{.i}), & i \neq j, \\ \mathrm{rdet}_k \mathbf{A}^{ii}, & i = j, \end{cases}$$

where $\mathbf{A}^{ii}_{.j}(\mathbf{a}_{.i})$ is obtained from $\mathbf{A}$ by replacing the jth column with the ith column, and then by deleting both the ith row and column, $k = \min\{I_n \setminus \{i\}\}$.

Lemma 2.2 *[4] Let L_{ij} be the left ij-th cofactor of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, that is, $\mathrm{cdet}_j \mathbf{A} = \sum\limits_{i=1}^{n} L_{ij} \cdot a_{ij}$ for all $j = 1, \ldots, n$. Then*

$$L_{ij} = \begin{cases} -\mathrm{cdet}_i \mathbf{A}_{i.}^{jj}(\mathbf{a}_{j.}), & i \neq j, \\ \mathrm{cdet}_k \mathbf{A}^{jj}, & i = j, \end{cases}$$

where $\mathbf{A}_{i.}^{jj}(\mathbf{a}_{j.})$ is obtained from $\mathbf{A}$ by replacing the ith row with the jth row, and then by deleting both the jth row and column, $k = \min\{J_n \setminus \{j\}\}$.

We recall some well-known definitions. The *conjugate* of a quaternion $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ is defined by $\overline{a} = a_o - a_1 i - a_2 j - a_3 k$. The *Hermitian adjoint matrix* of $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times m}$ is called the matrix $\mathbf{A}^* = \left(a_{ij}^*\right)_{m\times n}$ if $a_{ij}^* = \overline{a_{ji}}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times m}$ is *Hermitian* if $\mathbf{A}^* = \mathbf{A}$.

A following theorem has a key value in the theory of the column and row determinants.

Theorem 2.1 *[4] If $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is Hermitian, then $\mathrm{rdet}_1 \mathbf{A} = \cdots = \mathrm{rdet}_n \mathbf{A} = \mathrm{cdet}_1 \mathbf{A} = \cdots = \mathrm{cdet}_n \mathbf{A} \in \mathbb{R}$.*

Taking into account Theorem 2.1 we define the determinant of a Hermitian matrix by putting $\det \mathbf{A} := \mathrm{rdet}_i \mathbf{A} = \mathrm{cdet}_i \mathbf{A}$ for all $i = 1, \ldots, n$. This determinant of a Hermitian matrix coincides with the Moore determinant. The properties of the determinant of a Hermitian matrix are considered in [4] by means of the column and row determinants. Among them we note the followings.

Theorem 2.2 *[4] If the ith row of a Hermitian matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is replaced with a left linear combination of its other rows, i.e. $\mathbf{a}_{i.} = c_1 \mathbf{a}_{i_1.} + \cdots + c_k \mathbf{a}_{i_k.}$, where $c_l \in \mathbb{H}$ for all $l = 1, \ldots, k$ and $\{i, i_l\} \subset I_n$, then*

$$\mathrm{cdet}_i \mathbf{A}_{i.}(c_1 \cdot \mathbf{a}_{i_1.} + \cdots + c_k \cdot \mathbf{a}_{i_k.}) = \mathrm{rdet}_i \mathbf{A}_{i.}(c_1 \cdot \mathbf{a}_{i_1.} + \cdots + c_k \cdot \mathbf{a}_{i_k.}) = 0.$$

Theorem 2.3 *[4] If the jth column of a Hermitian matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is replaced with a right linear combination of its other columns, i.e. $\mathbf{a}_{.j} = \mathbf{a}_{.j_1} c_1 + \cdots + \mathbf{a}_{.j_k} c_k$, where $c_l \in \mathbb{H}$ for all $l = 1, \ldots, k$ and $\{j, j_l\} \subset J_n$, then*

$$\mathrm{cdet}_i \mathbf{A}_{.i}(\mathbf{a}_{.i_1} \cdot c_1 + \cdots + \mathbf{a}_{.i_k} \cdot c_k) = \mathrm{rdet}_i \mathbf{A}_{.i}(\mathbf{a}_{.i_1} \cdot c_1 + \cdots + \mathbf{a}_{.i_k} \cdot c_k) = 0.$$

The following theorem about determinantal representation of an inverse matrix of Hermitian follows immediately from these properties.

Theorem 2.4 *[4] If a Hermitian matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is such that* $\det \mathbf{A} \neq 0$*, then there exist a unique right inverse matrix* $(R\mathbf{A})^{-1}$ *and a unique left inverse matrix* $(L\mathbf{A})^{-1}$*, and* $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1} =: \mathbf{A}^{-1}$*. They possess the following determinantal representations:*

$$(R\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix}, \tag{1}$$

$$(L\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix}. \tag{2}$$

Here R_{ij}*,* L_{ij} *are right and left* ij*-th cofactors of* $\mathbf{A}$ *respectively for all* $i, j = 1, \ldots, n$*.*

To obtain determinantal representation of an arbitrary inverse matrix $\mathbf{A}^{-1}$, we consider the right $\mathbf{A}\mathbf{A}^*$ and left $\mathbf{A}^*\mathbf{A}$ corresponding Hermitian matrix.

Theorem 2.5 *[4] If an arbitrary column of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *is a right linear combination of its other columns, or an arbitrary row of* $\mathbf{A}^*$ *is a left linear combination of its others, then* $\det \mathbf{A}^*\mathbf{A} = 0$.

Since the principal submatrices of a Hermitian matrix are Hermitian, the principal minor may be defined as the determinant of its principal submatrix by analogy to the commutative case. We introduce *the rank by principal minors* that is the maximal order of a nonzero principal minor of a Hermitian matrix. The following theorem determines a relationship between it and the rank of a matrix defining as ceiling amount of right-linearly independent columns or left-linearly independent rows which form basis.

Theorem 2.6 *[4] A rank by principal minors of* $\mathbf{A}^*\mathbf{A}$ *is equal to its rank and a rank of* $\mathbf{A} \in \mathbb{H}^{m\times n}$*.*

Theorem 2.7 *[4] If $\mathbf{A} \in \mathbb{H}^{m\times n}$, then an arbitrary column of $\mathbf{A}$ is a right linear combination of its basis columns or an arbitrary row of $\mathbf{A}$ is a left linear combination of its basis rows.*

The criterion of singularity of a Hermitian matrix is obtained.

Theorem 2.8 *[4] The right-linearly independence of columns of $\mathbf{A} \in \mathbb{H}^{m\times n}$ or the left-linearly independence of rows of $\mathbf{A}^*$ is the necessary and sufficient condition for* $\det \mathbf{A}^*\mathbf{A} \neq 0$.

Theorem 2.9 *[4] If $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, then* $\det \mathbf{A}\mathbf{A}^* = \det \mathbf{A}^*\mathbf{A}$.

A concept of the double determinant is introduced by this theorem. This concept was initially introduced by L. Chen in [2].

Definition 2.3 *The determinant of the corresponding Hermitian matrix of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is called its double determinant, i.e.* $\mathrm{ddet}\mathbf{A} := \det(\mathbf{A}^*\mathbf{A}) = \det(\mathbf{A}\mathbf{A}^*)$.

The relationship between the double determinant and the noncommutative determinants of E. Moore, E. Study and J. Diedonne is obtained, $\mathrm{ddet}\mathbf{A} = \mathrm{Mdet}(\mathbf{A}^*\mathbf{A}) = \mathrm{Sdet}\mathbf{A} = \mathrm{Ddet}^2\mathbf{A}$. But unlike those, the double determinant can be expanded along an arbitrary row or column by means of the column and row determinants.

Definition 2.4 *Suppose $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$. We have a column expansion of* $\mathrm{ddet}\mathbf{A}$ *along the jth column,* $\mathrm{ddet}\mathbf{A} = \mathrm{cdet}_j(\mathbf{A}^*\mathbf{A}) = \sum\limits_i \mathbb{L}_{ij} \cdot a_{ij}$*, and a row expansion of it along the ith row,* $\mathrm{ddet}\mathbf{A} = \mathrm{rdet}_i(\mathbf{A}\mathbf{A}^*) = \sum\limits_j a_{ij} \cdot \mathbb{R}_{ij}$ *for all $i, j = 1, \ldots, n$. Then by definition of the left double ijth cofactor we put $\mathbb{L}_{ij}$ and by definition of the right double ijth cofactor we put $\mathbb{R}_{ij}$.*

Theorem 2.10 *[4] The necessary and sufficient condition of invertibility of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is* $\mathrm{ddet}\mathbf{A} \neq 0$. *Then there exists $\mathbf{A}^{-1} = (L\mathbf{A})^{-1} = (R\mathbf{A})^{-1}$, where*

$$(L\mathbf{A})^{-1} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^* = \frac{1}{\mathrm{ddet}\mathbf{A}} \begin{pmatrix} \mathbb{L}_{11} & \mathbb{L}_{21} & \ldots & \mathbb{L}_{n1} \\ \mathbb{L}_{12} & \mathbb{L}_{22} & \ldots & \mathbb{L}_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{L}_{1n} & \mathbb{L}_{2n} & \ldots & \mathbb{L}_{nn} \end{pmatrix}, \tag{3}$$

$$\left(R\mathbf{A}\right)^{-1} = \mathbf{A}^{*}\left(\mathbf{A}\mathbf{A}^{*}\right)^{-1} = \frac{1}{\mathrm{ddet}\mathbf{A}}\begin{pmatrix} \mathbb{R}_{11} & \mathbb{R}_{21} & \ldots & \mathbb{R}_{n1} \\ \mathbb{R}_{12} & \mathbb{R}_{22} & \ldots & \mathbb{R}_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{R}_{1n} & \mathbb{R}_{2n} & \ldots & \mathbb{R}_{nn} \end{pmatrix}, \tag{4}$$

and $\mathbb{L}_{ij} = \mathrm{cdet}_j(\mathbf{A}^{*}\mathbf{A})_{.j}\left(\mathbf{a}^{*}_{.i}\right)$, $\mathbb{R}_{ij} = \mathrm{rdet}_i(\mathbf{A}\mathbf{A}^{*})_{i.}\left(\mathbf{a}^{*}_{j.}\right)$ *for all* $i, j = 1, \ldots, n$.

This theorem introduces the determinantal representations of an inverse matrix by the left (1) and right (2) double cofactors. The inverse matrix $\mathbf{A}^{-1}$ of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ on the assumption of $\mathrm{ddet}\mathbf{A} \neq 0$ is represented by the analog of the classical adjoint matrix. If we denote this analog of the adjoint matrix over $\mathbb{H}$ by $\mathrm{Adj}[[\mathbf{A}]]$, where $\mathrm{Adj}[[\mathbf{A}]] = (\mathbb{L}_{ij})_{n\times n}$ or $\mathrm{Adj}[[\mathbf{A}]] = (\mathbb{R}_{ij})_{n\times n}$, then the following formula is valid over $\mathbb{H}$:

$$\mathbf{A}^{-1} = \frac{\mathrm{Adj}[[\mathbf{A}]]}{\mathrm{ddet}\mathbf{A}}.$$

Using the determinantal representations of an inverse matrix by the left (1) and right (2) analogs of a classical adjoint matrix we obtain the Cramer rule for right and left systems of linear equations respectively.

Theorem 2.11 *Let* $\mathbf{A}\cdot\mathbf{x} = \mathbf{y}$ *be a right system of linear equations with a matrix of coefficients* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, a column of constants* $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{H}^{n\times 1}$ *and a column of unknowns* $\mathbf{x} = (x_1, \ldots, x_n)^T$*. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then we have for all*$j = \overline{1, n}$

$$x_j = \frac{\mathrm{cdet}_j(\mathbf{A}^{*}\mathbf{A})_{.j}\left(\mathbf{f}\right)}{\mathrm{ddet}\mathbf{A}},$$

where $\mathbf{f} = \mathbf{A}^{*}\mathbf{y}$.

Theorem 2.12 *Let* $\mathbf{x}\cdot\mathbf{A} = \mathbf{y}$ *be a left system of linear equations with a matrix of coefficients* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, a row of constants* $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{H}^{1\times n}$ *and a row of unknowns* $\mathbf{x} = (x_1, \ldots, x_n)$*. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then we obtain for all* $i = \overline{1, n}$

$$x_i = \frac{\mathrm{rdet}_i(\mathbf{A}\mathbf{A}^{*})_{i.}\left(\mathbf{z}\right)}{\mathrm{ddet}\mathbf{A}},$$

where $\mathbf{z} = \mathbf{y}\mathbf{A}^{*}$.

3 Cramer's rule for some matrix equations.

We denote $\mathbf{A}^*\mathbf{B} =: \hat{\mathbf{B}} = (\hat{b}_{ij})$, $\mathbf{B}\mathbf{A}^* =: \check{\mathbf{B}} = (\check{b}_{ij})$.

Theorem 3.1 *Suppose*

$$\mathbf{A}\mathbf{X} = \mathbf{B} \tag{5}$$

is a right matrix equation, where $\{\mathbf{A}, \mathbf{B}\} \in \mathrm{M}(n, \mathbb{H})$ *are given,* $\mathbf{X} \in \mathrm{M}(n, \mathbb{H})$ *is unknown. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then (5) has a unique solution, and the solution is*

$$x_{i\,j} = \frac{\mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\hat{\mathbf{b}}_{.j}\right)}{\mathrm{ddet}\mathbf{A}} \tag{6}$$

where $\hat{\mathbf{b}}_{.j}$ *is the* j*th column of* $\hat{\mathbf{B}}$ *for all* $i, j = 1, ..., n$*.*

Proof. By Theorem 2.10 the matrix $\mathbf{A}$ is invertible. There exists the unique inverse matrix $\mathbf{A}^{-1}$. From this it follows that the solution of (5) exists and is unique, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. If we represent $\mathbf{A}^{-1} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$ as a left inverse by (3), and use the determinantal representation of $(\mathbf{A}^*\mathbf{A})^{-1}$ by (2), then for all $i, j = 1, ..., n$ we obtain

$$x_{ij} = \frac{1}{\mathrm{ddet}\mathbf{A}} \sum_{k=1}^{n} L_{ki}\hat{b}_{kj},$$

where L_{ij} is a left ijth cofactor of $(\mathbf{A}^*\mathbf{A})$ for all $i, j = 1, ..., n$. From this by Lemma 2.2 and denoting the j-th column of $\hat{\mathbf{B}}$ by $\hat{\mathbf{b}}_{.j}$, it follows (6).■

Theorem 3.2 *Suppose*

$$\mathbf{X}\mathbf{A} = \mathbf{B} \tag{7}$$

is a left matrix equation, where $\{\mathbf{A}, \mathbf{B}\} \in \mathrm{M}(n, \mathbb{H})$ *are given,* $\mathbf{X} \in \mathrm{M}(n, \mathbb{H})$ *is unknown. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then (7) has a unique solution, and the solution is*

$$x_{i\,j} = \frac{\mathrm{rdet}_j(\mathbf{A}\mathbf{A}^*)_{j.}\left(\check{\mathbf{b}}_{i.}\right)}{\mathrm{ddet}\mathbf{A}} \tag{8}$$

where $\check{\mathbf{b}}_{i.}$ *is the* i*th column of* $\check{\mathbf{B}}$ *for all* $i, j = 1, ..., n$*.*

Proof. By Theorem 2.10 the matrix $\mathbf{A}$ is invertible. There exists the unique inverse matrix $\mathbf{A}^{-1}$. From this it follows that the solution of (7) exists and is unique, $\mathbf{X} = \mathbf{B}\mathbf{A}^{-1}$. If we represent $(\mathbf{A})^{-1} = \mathbf{A}^*(\mathbf{A}\mathbf{A}^*)^{-1}$ as a right inverse

by (4) and use the determinantal representation of $(\mathbf{A}\mathbf{A}^*)^{-1}$ by (1), then for all $i, j = 1, ..., n$ we have

$$x_{ij} = \frac{1}{\mathrm{ddet}\mathbf{A}} \sum_{k=1}^{n} \check{b}_{i\,k} R_{jk}.$$

where R_{ij} is a right ijth cofactor of $(\mathbf{A}\mathbf{A}^*)$ for all $i, j = 1, ..., n$. From this by means of Lemma 2.1 and denoting the ith row of $\check{\mathbf{B}}$ by $\check{\mathbf{b}}_{i\,.}$, it follows (8). ■

We denote $\mathbf{A}^*\mathbf{C}\mathbf{B}^* =: \tilde{\mathbf{C}} = (\tilde{c}_{ij})$.

Theorem 3.3 *Suppose*

$$\mathbf{AXB} = \mathbf{C} \tag{9}$$

is a two-sided matrix equation, where $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\} \in \mathrm{M}(n, \mathbb{H})$ *are given,* $\mathbf{X} \in \mathrm{M}(n, \mathbb{H})$ *is unknown. If* $\mathrm{ddet}\mathbf{A} \neq 0$ *and* $\mathrm{ddet}\mathbf{B} \neq 0$ *, then (9) has a unique solution, and the solution is*

$$x_{i\,j} = \frac{\mathrm{rdet}_j (\mathbf{B}\mathbf{B}^*)_{j.} \left(\mathbf{c}_{i.}^{\mathbf{A}}\right)}{\mathrm{ddet}\mathbf{A} \cdot \mathrm{ddet}\mathbf{B}}, \tag{10}$$

or

$$x_{i\,j} = \frac{\mathrm{cdet}_i (\mathbf{A}^*\mathbf{A})_{.\,i} \left(\mathbf{c}_{.j}^{\mathbf{B}}\right)}{\mathrm{ddet}\mathbf{A} \cdot \mathrm{ddet}\mathbf{B}}, \tag{11}$$

where $\mathbf{c}_{i.}^{\mathbf{A}} := \left(\mathrm{cdet}_i (\mathbf{A}^*\mathbf{A})_{.\,i} \left(\tilde{\mathbf{c}}_{.1}\right), \ldots, \mathrm{cdet}_i (\mathbf{A}^*\mathbf{A})_{.\,i} \left(\tilde{\mathbf{c}}_{.n}\right)\right)$ *is the row vector and* $\mathbf{c}_{.j}^{\mathbf{B}} := \left(\mathrm{rdet}_j (\mathbf{B}\mathbf{B}^*)_{j.} \left(\tilde{\mathbf{c}}_{1\,.}\right), \ldots, \mathrm{rdet}_j (\mathbf{B}\mathbf{B}^*)_{j.} \left(\tilde{\mathbf{c}}_{n\,.}\right)\right)^T$ *is the column vector and* $\tilde{\mathbf{c}}_{i\,.}$, $\tilde{\mathbf{c}}_{.\,j}$ *are the ith row vector and the jth column vector of* $\widetilde{\mathbf{C}}$*, respectively, for all* $i, j = 1, ..., n$.

Proof. By Theorem 2.10 the matrices $\mathbf{A}$ and $\mathbf{B}$ are invertible. There exist the unique inverse matrices $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$. From this it follows that the solution of (9) exists and is unique, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$. If we represent $\mathbf{A}^{-1} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$ as a left inverse and $(\mathbf{B})^{-1} = \mathbf{B}^* (\mathbf{B}\mathbf{B}^*)^{-1}$ as a right inverse, then for all

$i,j=1,...,n$ we have

$$\mathbf{X}=\left(\mathbf{A}^{*}\mathbf{A}\right)^{-1}\mathbf{A}^{*}\mathbf{C}\mathbf{B}^{*}\left(\mathbf{B}\mathbf{B}^{*}\right)^{-1}=$$
$$=\begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n}\\ x_{21} & x_{22} & \ldots & x_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ x_{n1} & x_{n2} & \ldots & x_{nn}\end{pmatrix}=\tfrac{1}{\mathrm{ddet}\mathbf{A}}\begin{pmatrix} L_{11}^{\mathbf{A}} & L_{21}^{\mathbf{A}} & \ldots & L_{n1}^{\mathbf{A}}\\ L_{12}^{\mathbf{A}} & L_{22}^{\mathbf{A}} & \ldots & L_{n2}^{\mathbf{A}}\\ \ldots & \ldots & \ldots & \ldots\\ L_{1n}^{\mathbf{A}} & L_{2n}^{\mathbf{A}} & \ldots & L_{nn}^{\mathbf{A}}\end{pmatrix}\times$$
$$\times\begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \ldots & \tilde{c}_{1n}\\ \tilde{c}_{21} & \tilde{c}_{22} & \ldots & \tilde{c}_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ \tilde{c}_{n1} & \tilde{c}_{n2} & \ldots & \tilde{c}_{nn}\end{pmatrix}\tfrac{1}{\mathrm{ddet}\mathbf{A}}\begin{pmatrix} R_{11}^{\mathbf{B}} & R_{21}^{\mathbf{B}} & \ldots & R_{n1}^{\mathbf{B}}\\ R_{12}^{\mathbf{B}} & R_{22}^{\mathbf{B}} & \ldots & R_{n2}^{\mathbf{B}}\\ \ldots & \ldots & \ldots & \ldots\\ R_{1n}^{\mathbf{B}} & R_{2n}^{\mathbf{B}} & \ldots & R_{nn}^{\mathbf{B}}\end{pmatrix},$$

where $L_{ij}^{\mathbf{A}}$ is a left ijth cofactor of $(\mathbf{A}^{*}\mathbf{A})$ and $R_{ij}^{\mathbf{B}}$ is a right ijth cofactor of $(\mathbf{B}\mathbf{B}^{*})$ for all $i,j=1,...,n$. This implies

$$x_{ij}=\frac{\sum\limits_{m=1}^{n}\left(\sum\limits_{k=1}^{n}L_{ki}^{\mathbf{A}}\tilde{c}_{km}\right)R_{jm}^{\mathbf{B}}}{\mathrm{ddet}\mathbf{A}\cdot\mathrm{ddet}\mathbf{B}},\tag{12}$$

for all $i,j=\overline{1,n}$. From this by Lemma 2.2, we obtain

$$\sum_{k=1}^{n}L_{ki}^{\mathbf{A}}\tilde{c}_{k\,m}=\mathrm{cdet}_{i}(\mathbf{A}^{*}\mathbf{A})_{.\,i}\left(\tilde{\mathbf{c}}_{.\,m}\right),$$

where $\tilde{\mathbf{c}}_{.\,m}$ is the mth column-vector of $\tilde{\mathbf{C}}$ for all $m=1,...,n$. Denote by $\mathbf{c}_{i\,.}^{\mathbf{A}}:=\left(\mathrm{cdet}_{i}(\mathbf{A}^{*}\mathbf{A})_{.\,i}\left(\tilde{\mathbf{c}}_{.1}\right),\ldots,\mathrm{cdet}_{i}(\mathbf{A}^{*}\mathbf{A})_{.\,i}\left(\tilde{\mathbf{c}}_{.n}\right)\right)$ the row-vector for all $i=1,...,n$. Reducing the sum $\sum\limits_{m=1}^{n}\left(\sum\limits_{k=1}^{n}L_{ki}^{\mathbf{A}}\tilde{c}_{km}\right)R_{jm}^{\mathbf{B}}$ by Lemma 2.1, we obtain an analog of Cramer's rule for (9) by (10).

Having changed the order of summation in (12), we have

$$x_{ij}=\frac{\sum\limits_{k=1}^{n}L_{ki}^{\mathbf{A}}\left(\sum\limits_{m=1}^{n}\tilde{c}_{km}R_{jm}^{\mathbf{B}}\right)}{\mathrm{ddet}\mathbf{A}\cdot\mathrm{ddet}\mathbf{B}}.$$

By Lemma 2.1, we obtain $\sum\limits_{m=1}^{n}c_{k\,m}R_{j\,m}^{\mathbf{B}}=\mathrm{rdet}_{j}(\mathbf{B}\mathbf{B}^{*})_{j.}\left(\tilde{\mathbf{c}}_{k\,.}\right)$, where $\tilde{\mathbf{c}}_{k\,.}$ is a kth row-vector of $\tilde{\mathbf{C}}$ for all $k=1,...,n$. We denote by

$$\mathbf{c}_{.\,j}^{\mathbf{B}}:=\left(\mathrm{rdet}_{j}(\mathbf{B}\mathbf{B}^{*})_{j.}\left(\tilde{\mathbf{c}}_{1\,.}\right),\ldots,\mathrm{rdet}_{j}(\mathbf{B}\mathbf{B}^{*})_{j.}\left(\tilde{\mathbf{c}}_{n\,.}\right)\right)^{T}$$

the column-vector for all $j = 1, ..., n$. Reducing the sum $\sum\limits_{k=1}^{n} L_{ki}^{\mathbf{A}} \left(\sum\limits_{m=1}^{n} \tilde{c}_{km} R_{jm}^{\mathbf{B}} \right)$ by Lemma 2.2, we obtain Cramer's rule for (9) by (11). ■

In solving the matrix equations by Cramer's rules (6), (8), (10), (11) we do not use the complex representation of quaternion matrices and work only in the quaternion skew field.

4 Example

Let us consider the two-sided matrix equation

$$\mathbf{AXB} = \mathbf{C} \tag{13}$$

where $\mathbf{A} = \begin{pmatrix} i & -j & k \\ k & -i & 1 \\ 2 & k & -j \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -k & j & 2 \\ i & k & i \\ -j & 1 & i \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & i & j \\ k & j & -2 \\ i & 1 & j \end{pmatrix}$. Then we have $\mathbf{A}^* = \begin{pmatrix} -i & -k & 2 \\ j & i & -k \\ -k & 1 & j \end{pmatrix}$, $\mathbf{A}^*\mathbf{A} = \begin{pmatrix} 6 & j+3k & -j-k \\ -j-3k & 3 & i \\ j+k & -i & 3 \end{pmatrix}$ and

$$\mathbf{B}^* = \begin{pmatrix} k & -i & j \\ -j & -k & 1 \\ 2 & -i & -i \end{pmatrix}, \mathbf{BB}^* = \begin{pmatrix} 6 & -3i+j & -i+j \\ 3i-j & 3 & 1+2k \\ i-j & 1-2k & 3 \end{pmatrix},$$

$\tilde{\mathbf{C}} = \mathbf{A}^*\mathbf{C}\mathbf{B}^* = \begin{pmatrix} 2k & 1-i-k & 3+i+3k \\ -2-4i & -2+i-k & i-k \\ -4+2i & 1+2i+j & 1+4i+j \end{pmatrix}$. It is easy to get, $\mathrm{ddet}\mathbf{A} = \det \mathbf{A}^*\mathbf{A} = 8$ and $\mathrm{ddet}\mathbf{B} = \det \mathbf{BB}^* = 4$. Therefore (13) has a solution. We shall find it by (10). At first we obtain the row-vectors $\mathbf{c}_{i.}^{\mathbf{A}}$ for all $i = 1, 2, 3$.

$$\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}\left(\tilde{\mathbf{c}}_{.1}\right) = \mathrm{cdet}_1 \begin{pmatrix} 2k & j+3k & -j-k \\ -2-4i & 3 & i \\ -4+2i & -i & 3 \end{pmatrix} = 3 \cdot 3(2k) -$$
$$-i(-i)(3j+5k) + (-j-k)(-i)(-2-4i) - 3(j+3k)(-2-4i) +$$
$$+(j+3k)i(-4+2i) - 3(-j-k)(-4+2i) =$$
$$= 24j + 8k,$$

and so forth. Continuing in the same way, we get

$$\begin{gathered}\mathbf{c}_{1.}^{\mathbf{A}}=(24j+8k,-8-8i+4j+4k,8+8i+4j+4k),\\ \mathbf{c}_{2.}^{\mathbf{A}}=(-20-36i,-10-2i-12j-12k,-2-2i+12j+4k),\\ \mathbf{c}_{3.}^{\mathbf{A}}=(12+4i,6+2i+12j-4k,6+10i-4j+4k).\end{gathered}$$

Then by (10) we have

$$x_{1\,1}=\frac{\mathrm{rdet}_1(\mathbf{B}\mathbf{B}^*)_{1.}\left(\mathbf{c}_{1.}^{\mathbf{A}}\right)}{\mathrm{ddet}\mathbf{A}\cdot\mathrm{ddet}\mathbf{B}}=$$

$$\begin{gathered}=\frac{1}{32}\cdot\mathrm{rdet}_1\begin{pmatrix}24j+8k & -8-8i+4j+4k & 8+8i+4j+4k\\ 3i-j & 3 & 1+2k\\ i-j & 1-2k & 3\end{pmatrix}=\\ =\frac{1}{30}\cdot((24j+8k)\cdot3\cdot3-(24j+8k)(1+2k)(1-2k)+\\ +(-8-8i+4j+4k)(1+2k)(i-j)-(-8-8i+4j+4k)(3i-j)3+\\ +(8+8i+4j+4k)(1-2k)(3i-j)-(8+8i+4j+4k)(i-j)3)=\\ =\frac{1}{32}\cdot(-32+32i),\end{gathered}$$

and so forth. Continuing in the same way, we obtain

$$\begin{gathered}x_{11}=\frac{-32+32i}{32},x_{12}=\frac{-88-72i+24j-8k}{32},x_{13}=\frac{24+8i-40j+56k}{32},\\ x_{21}=\frac{-16i+32j-48k}{32},x_{22}=\frac{20-28i-116j-76k}{32},x_{23}=\frac{-44+68i+20j+12k}{32},\\ x_{31}=\frac{16+16j+32k}{32},x_{32}=\frac{20+44i+52j-28k}{32},x_{32}=\frac{-12-20i+12j-4k}{32}.\end{gathered}$$

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